Lecture 7, 10/12/12

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Determinants

Goal

- Define a function that
  - Associates a real number with a square matrix
  - Tell us if that matrix is invertible.

Square Matrix ⟶ Number

$$A \xrightarrow{\det} \det(A)$$

Goal

Square Matrix ⟶ Number

A $\xrightarrow{\text{det}}$ det(A)

- Define this in such a way that

det(A)=0 ⟷ Singular

det(A)≠0 ⟷ Non-Singular

Cases

- We'll define this in cases
  - 1x1 matrices
  - 2x2 matrices
  - 3x3 matrices
  - nxn matrices (rarely feasible by hand)

Case 1 (1x1 matrix)

- A 1x1 matrix is really just a number.

$$A = [a]$$

- Define: $det(A) = a$
- Then

$$\det(A)=0 \longleftrightarrow \text{Singular}$$

$$\det(A)\neq 0 \longleftrightarrow \text{Non-Singular}$$

Case 2 (2x2 matrix)

- Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

- Row reduction leads to

$$A = \begin{bmatrix} a_{11} & a_{12} \\ 0 & \boxed{a_{11}a_{22} - a_{21}a_{12}} \end{bmatrix}$$

- Define $det(A) = a_{11}a_{22} - a_{21}a_{12}$

Case 2 (2x2 matrix)

- With this definition, if $a_{11} \neq 0$, then:

$$\det(A)=0 \longleftrightarrow \text{Singular}$$

$$\det(A)\neq 0 \longleftrightarrow \text{Non-Singular}$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{11}a_{22} - a_{21}a_{12} \end{bmatrix}$$

$$det(A) = a_{11}a_{22} - a_{21}a_{12}$$

Geometric Definition

- The determinant is the product of the blue diagonal minus the product of the red diagonal

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}$$

Example

$$A = \begin{bmatrix} 1 & -4 \\ 6 & 3 \end{bmatrix}$$

$$det(A) = 1 \cdot 3 - (-4) \cdot 6 = 27$$

Non-Singular

$$A = \begin{bmatrix} 2 & 7 \\ 4 & 14 \end{bmatrix}$$

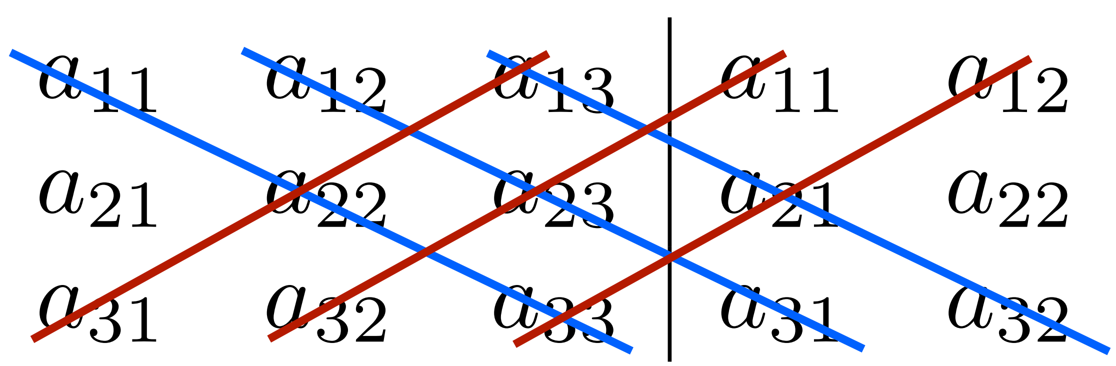
$$det(A) = 2 \cdot 14 - 7 \cdot 4 = 0$$

Singular

Case 3 (3x3 matrix)

- Geometric definition (not in book)
- Blue diagonals minus red diagonals

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$



Case 3 (3x3 matrix)

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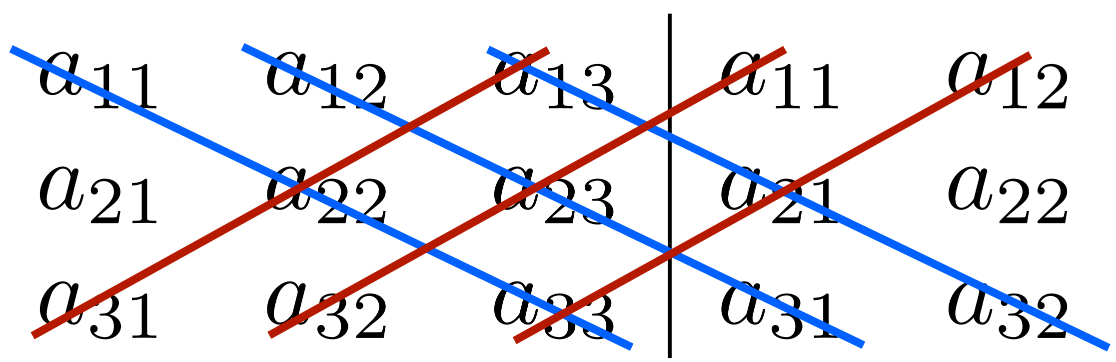

$$det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{31}a_{21}a_{32}$$
$$-a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

Example

$$det(A) = det\left(\begin{bmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 0 \end{bmatrix}\right) = 240$$

- So ‘A’ is non-singular (ie. it has an inverse).

Example

$$det(A) = det\left(\begin{bmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

- We know 'A' is not invertible.
- How? If it were invertible then $A\vec{x} = \vec{b}$ would have a solution.
- But it doesn't since this is under-constrained.

Example

$$det(A) = det\left(\begin{bmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 0 & 0 & 0 \end{bmatrix}\right) = 0$$

- Alternatively, you can just take the determinant and see that 'A' is singular (ie. it has no inverse).

Similarly

$$det(A) = det\left(\begin{bmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}\right) = 0$$

- 'A' is singular (ie. it has no inverse).
- This makes sense since this is representative of an under-constrained coefficient matrix.

IMPORTANT

- This method of looking at diagonals DOES NOT WORK beyond the 3x3 case.
- For 4x4 and up, you have to use ***expansion by minors*** also referred to as ***cofactor expansion.***